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On the Generalized Bertrand Curves in Euclidean N -spaces

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Abstract. In this article, we give a necessary condition for a C^∞ -special Frenet curve in \mathbb{R}^N being a generalized Bertrand curve.

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1 Introduction

A curve $C : I \rightarrow \mathbb{R}^N$ is called a C^∞ -special Frenet curve in \mathbb{R}^N if it is a smooth and regular curve with well-defined Frenet frames $\{t, n_1, \dots, n_{N-1}\}$ and non-zero curvatures (i.e., curvatures never vanish at any point of curves) along the curve. On the other hand, in literature, a C^∞ -special Frenet curve in \mathbb{R}^3 is called a Bertrand curve if there exists a distinct curve $\bar{C}(s) = C(s) + r(s) \cdot n_1(s)$, where n_1 is well-defined along C , such that the 1-normal lines of $C(s)$ and $\bar{C}(s)$ are equal for all $s \in I$. Furthermore, \bar{C} is called the Bertrand mate of C .

Bertrand curves in \mathbb{R}^3 have many interesting geometric properties (e.g., see p.26 in [1] for more details). These types of curves have also been applied in computer-aided geometric design (CAGD) (e.g., see [5], [7]). In [3] Hayden has suggested to extend the definition of Bertrand curves in \mathbb{R}^3 to those in \mathbb{R}^N or Riemannian manifolds. However, Pears in [6] showed that a Bertrand curve in \mathbb{R}^N must belong to a three-dimensional subspace in \mathbb{R}^N , since its curvatures of higher order must be identically equal to zero, i.e., $k_j = 0$ for all $j \geq 3$. This implies that a Bertrand curve in \mathbb{R}^N can't be a C^∞ -special Frenet curve in \mathbb{R}^N . Notice that a C^∞ -special Frenet curve in \mathbb{R}^N can not be confined in

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a lower dimensional subspace of \mathbb{R}^N because all of its curvatures are nonzero. Thus the classical definition of Bertrand curves is not suitable for C^∞ -special Frenet curves in \mathbb{R}^N . This phenomenon motivates us to extend the notion of Bertrand curves to C^∞ -special Frenet curves in \mathbb{R}^N when $N \geq 4$. To the authors' knowledge, [4] is the only article extending the notion of Bertrand curves to Euclidean spaces of higher dimensions. The reader is referred to [4] for explicit examples of the so-called $(1, 3)$ -Bertrand curves in \mathbb{R}^4 (which is a type of generalized Bertrand curves).

In this article, we generalize the definition of classical Bertrand curves in \mathbb{R}^3 and $(1, 3)$ -Bertrand curves in \mathbb{R}^4 (see [4]) by defining the so-called generalized Bertrand curves for the class of C^∞ -special Frenet curves in \mathbb{R}^N , where $N \geq 4$ (see Definition 1). Our main result gives a necessary condition for existence of the generalized Bertrand curve in \mathbb{R}^N . We found that only a particular type of generalized Bertrand curves exists in \mathbb{R}^N .

1 Definition (The Generalized Bertrand Curves). Assume $C : I \rightarrow \mathbb{R}^N$ is a C^∞ -special Frenet curve. Let $i_p \in \{1, 2, \dots, N-1\}$, where $p \in \{1, \dots, m\}$ and $m \in \{1, 2, \dots, N-1\}$. Denote by n_{i_p} the i_p -th unit normal vector field of the curve C . Then, the curve C is called a (i_1, \dots, i_m) -Bertrand curve if there exists a distinct C^∞ -special Frenet curve,

$$\bar{C}(s) = C(s) + \sum_{p=1}^m \alpha_{i_p}(s) \cdot n_{i_p}(s), \quad (1)$$

such that the Frenet (i_1, \dots, i_m) -normal planes at $C(s)$ and $\bar{C}(s)$ coincide for all $s \in I$.

For our convenience, we call a (i_1, \dots, i_m) -Bertrand curve the generalized Bertrand curves, and we always let $1 \leq i_1 < i_2 < \dots < i_m \leq N-1$.

2 Theorem. *If a C^∞ -special Frenet curve in \mathbb{R}^N is a generalized Bertrand curve, then it must be the type of $(1, i_2, \dots, i_m)$ -Bertrand curve.*

3 Remark. The generalized Bertrand curves still keep certain geometric properties. For example, by a straightforward computation, one can verify that the distance between a generalized Bertrand curve and its mate (offset) along the curve remains constant. For geometric properties of $(1, 3)$ -Bertrand curves in \mathbb{R}^4 , the reader is referred to [4]. For generalized Bertrand curves in \mathbb{R}^N , we leave the discussion to our future work.

2 Proofs

We will argue by contradiction. Namely, we assume that the C^∞ -special Frenet curve C in \mathbb{R}^N is a (i_1, \dots, i_m) -Bertrand curve with $i_1 \geq 2$, then a

contradiction would happen.

Denote by $\bar{s} = \varphi(s)$ the arc-length parameter of \bar{C} . Let

$$K_{j,i}(s) = -k_i(s)\delta_j^{i-1} + k_{i+1}(s)\delta_j^{i+1}, \quad (2)$$

where δ_j^i is the Kronecker's delta and k_i are higher curvatures of \mathbb{R}^N if $i \in \{1, \dots, N-1\}$; otherwise $k_i = 0$ (see [2]). Then the Frenet equations can be written as

$$n'_i(s) = \sum_{j=0}^{N-1} K_{j,i}(s)n_j(s), \quad (3)$$

where $n_0(s) = t(s)$ and $i \in \{0, \dots, N-1\}$. By differentiating (1) with respect to s , we obtain

$$\begin{aligned} \varphi'(s) \cdot \bar{n}_0(s) &= \bar{C}'(s) \\ &= n_0(s) + \sum_{p=1}^m \alpha'_{i_p}(s) \cdot n_{i_p}(s) + \sum_{j=0}^{N-1} \sum_{p=1}^m K_{j,i_p}(s) \cdot \alpha_{i_p}(s) \cdot n_j(s) \\ &= \sum_{j=0}^{N-1} \beta_j(s)n_j(s), \end{aligned} \quad (4)$$

where n_0 and \bar{n}_0 denote the unit tangent vectors of C and \bar{C} respectively. Since by assumption the normal plane spanned by $\bar{n}_{i_1}(s), \dots, \bar{n}_{i_m}(s)$ coincides with the one spanned by $n_{i_1}(s), \dots, n_{i_m}(s)$, there exists a matrix $T(s) \in O(m)$ such that

$$(\bar{n}_{i_1}(s), \dots, \bar{n}_{i_m}(s))^t = T(s)(n_{i_1}(s), \dots, n_{i_m}(s))^t,$$

for all $s \in I$. In other words,

$$\bar{n}_{i_q}(s) = \sum_{p=1}^m T_{qp}(s)n_{i_p}(s), \quad (5)$$

where T_{qp} is the (q, p) -th entry of the matrix T . Thus by (4) and (5), we have

$$\begin{aligned} 0 &= \langle \varphi'(s) \cdot \bar{n}_0(s), \bar{n}_{i_q}(s) \rangle = \sum_{p=1}^m T_{qp}(s) \langle \sum_{j=0}^{N-1} \beta_j(s) \cdot n_j(s), n_{i_p}(s) \rangle \\ &= \sum_{p=1}^m T_{qp}(s) \beta_{i_p}(s), \end{aligned} \quad (6)$$

for each fixed $q \in \{1, 2, \dots, m\}$. Since $\det T(s) = \pm 1 \neq 0$, it follows from (4) and (6) that

$$0 = \beta_{i_q}(s) = \alpha'_{i_q}(s) + \sum_{p=1}^m K_{i_q, i_p}(s) \cdot \alpha_{i_p}(s), \quad (7)$$

for each $q \in \{1, 2, \dots, m\}$. By (4),

$$\begin{aligned} \bar{n}_0(s) &= \frac{1}{\varphi'(s)} n_0(s) + \sum_{p=1}^m \frac{1}{\varphi'(s)} \alpha'_{i_p}(s) \cdot n_{i_p}(s) \\ &\quad + \sum_{j=0}^{N-1} \sum_{p=1}^m \frac{1}{\varphi'(s)} K_{j, i_p}(s) \cdot \alpha_{i_p}(s) \cdot n_j(s). \end{aligned} \quad (8)$$

By differentiating (8) with respect to s , we obtain

$$\begin{aligned} &\varphi'(s) \cdot \bar{k}_1(s) \cdot \bar{n}_1(s) \\ &= \left(\frac{1}{\varphi'(s)} \right)' n_0(s) + \frac{1}{\varphi'(s)} k_1(s) n_1(s) + \sum_{p=1}^m \left(\frac{1}{\varphi'(s)} \alpha'_{i_p}(s) \right)' n_{i_p}(s) \\ &\quad + \sum_{j=1}^{N-1} \left(\sum_{p=1}^m \frac{1}{\varphi'(s)} K_{j, i_p}(s) \cdot \alpha'_{i_p}(s) \right)' \cdot n_j(s) \\ &\quad + \sum_{j=1}^{N-1} \left(\sum_{p=1}^m \frac{1}{\varphi'(s)} K_{j, i_p}(s) \cdot \alpha_{i_p}(s) \right)' \cdot n_j(s) \\ &\quad + \sum_{j=0}^{N-1} \left(\sum_{p=1}^m \frac{1}{\varphi'(s)} [K_{j-1, i_p}(s) \cdot k_j(s) - K_{j+1, i_p}(s) \cdot k_{j+1}(s)] \cdot \alpha_{i_p}(s) \right)' \cdot n_j(s) \\ &= \sum_{j=0}^{N-1} \gamma_j(s) \cdot n_j(s), \end{aligned} \quad (9)$$

where $k_N = 0$. By (9), (5) and assuming $i_q \geq 2$ for all $q \in \{1, 2, \dots, m\}$, we have

$$0 = \langle \varphi'(s) \bar{k}_1(\varphi(s)) \bar{n}_1(\varphi(s)), \bar{n}_{i_q}(\varphi(s)) \rangle = \sum_{p=1}^m T_{qp}(s) \gamma_{i_p}(s). \quad (10)$$

Since $\det T(s) = \pm 1 \neq 0$, it follows from (9) and (10) that

$$\begin{aligned}
 0 &= \gamma_{i_q}(s) \\
 &= \left(\frac{1}{\varphi'(s)} \alpha'_{i_q}(s) \right)' + \sum_{p=1}^m \frac{1}{\varphi'(s)} K_{i_q, i_p}(s) \alpha'_{i_p}(s) \\
 &\quad + \sum_{p=1}^m \left(\frac{1}{\varphi'(s)} K_{i_q, i_p}(s) \alpha_{i_p}(s) \right)' \\
 &\quad + \sum_{p=1}^m \frac{1}{\varphi'(s)} (K_{-1+i_q, i_p}(s) k_{i_q} - K_{1+i_q, i_p}(s) k_{1+i_q}) \alpha_{i_p}(s),
 \end{aligned} \tag{11}$$

for all $q \in \{1, 2, \dots, m\}$. Below we omit the arc-length parameter s of C without confusion. Denote by $A = (\alpha_{i_1}, \dots, \alpha_{i_m})^t$, and let $B = (B_{lp})$, and $R = (R_{lp})$ to be

$$B_{lp} = K_{i_l, i_p}, \tag{12}$$

$$R_{lp} = K_{-1+i_l, i_p} k_{i_l} - K_{1+i_l, i_p} k_{1+i_l}. \tag{13}$$

Then (7) and (11) can be written respectively as

$$A' + BA = 0, \tag{14}$$

$$\left(\frac{1}{\varphi'} A' \right)' + \frac{1}{\varphi'} BA' + \left(\frac{1}{\varphi'} BA \right)' + \frac{1}{\varphi'} RA = 0. \tag{15}$$

Substituting A' by $-BA$ in (15), we can simplify (15) as

$$(R - B^2)A = 0. \tag{16}$$

4 Lemma. *The $m \times m$ matrix $R - B^2$ is symmetric and can be written as*

$$\begin{pmatrix}
 D_1 + F_1 & N_1 & 0 & \cdots & \cdots & 0 \\
 N_1 & D_2 + F_2 & N_2 & \ddots & \ddots & \vdots \\
 0 & N_2 & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & N_{m-2} & 0 \\
 \vdots & \ddots & \ddots & N_{m-2} & D_{m-1} + F_{m-1} & N_{m-1} \\
 0 & \cdots & \cdots & 0 & N_{m-1} & D_m + F_m
 \end{pmatrix},$$

where

$$\begin{cases}
 D_q = K_{i_{q-1}, i_q}^2 - k_{i_q}^2, \\
 F_q = K_{i_q, i_{q+1}}^2 - k_{1+i_q}^2, \\
 N_q = K_{-1+i_{q+1}, i_q} k_{i_q},
 \end{cases} \tag{17}$$

and we let $K_{i_p, i_q} = 0$, if i_p or i_q is not defined.

PROOF. From (12), it is obvious that $B^t = -B$, thus B^2 is symmetric. By applying (13), (12) and (2), it is easy to verify that the matrix R is symmetric and to compute all entries of $R - B^2$. We leave it to the reader. \square

We can decompose the matrix $R - B^2$ into a sum of matrices. Namely,

$$\begin{aligned}
 R - B^2 &= \sum_{q=1}^{m+1} E_q \\
 &= \begin{pmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & F_m \end{pmatrix} \\
 &\quad + \sum_{q=2}^m \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & F_{q-1} & N_{q-1} & \vdots & \vdots \\ \vdots & \vdots & N_{q-1} & D_q & \vdots & \vdots \\ \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.
 \end{aligned}$$

5 Lemma. For each fixed $q \in \{1, \dots, m+1\}$ and $X = (x_1, \dots, x_m)^t$,

$$\langle E_q X, X \rangle \leq 0. \quad (18)$$

PROOF. It is easy to verify (18) by using (17) and (2). We leave it to the reader. \square

Observe that Lemma 5 and (16) imply

$$\langle E_q A, A \rangle = 0,$$

for each fixed q .

6 Lemma. (i) Assume $i_{q+1} - i_q \geq 3$. Then, $\alpha_{i_q} = 0 = \alpha_{i_{q+1}}$.

(ii) Assume $i_{q+1} - i_q = 2$. Then, $\alpha_{i_q} = 0$ if and only if $\alpha_{i_{q+1}} = 0$.

(iii) Assume $i_{q+1} - i_q = 1$. Then, $\alpha_{i_{q-1}} = 0 = \alpha_{i_q}$ implies $\alpha_{i_{q+1}} = 0$, where we set $\alpha_{i_0} = 0$.

PROOF. **Case (i):** $i_{q+1} - i_q \geq 3$. From

$$0 = \langle E_{q+1}A, A \rangle = -[(k_{1+i_q}\alpha_{i_q})^2 + (k_{i_{q+1}}\alpha_{i_{q+1}})^2],$$

it follows that

$$k_{1+i_q}\alpha_{i_q} = 0 = k_{i_{q+1}}\alpha_{i_{q+1}}.$$

Thus $\alpha_{i_q} = 0 = \alpha_{i_{q+1}}$.

Case (ii): $i_{q+1} - i_q = 2$. From

$$0 = \langle E_{q+1}A, A \rangle = -(k_{1+i_q}\alpha_{i_q} - k_{i_{q+1}}\alpha_{i_{q+1}})^2,$$

it follows that

$$k_{1+i_q}\alpha_{i_q} = k_{i_{q+1}}\alpha_{i_{q+1}}.$$

Thus $\alpha_{i_q} = 0$ if and only if $\alpha_{i_{q+1}} = 0$.

Case (iii): $i_{q+1} - i_q = 1$. By (7), we have

$$-\alpha'_{i_q} = K_{i_q, i_{q-1}}\alpha_{i_{q-1}} - k_{i_{q+1}}\alpha_{i_{q+1}}.$$

By assuming $\alpha_{i_{q-1}} = 0 = \alpha_{i_q}$, it follows that $\alpha_{i_{q+1}} = 0$.

QED

PROOF OF THEOREM 2. By $\langle E_1A, A \rangle = 0$, we obtain $k_{i_1}^2\alpha_{i_1}^2 = 0$. Hence, $\alpha_{i_1} = 0$. Then, by applying Lemma 6 inductively, we obtain $\alpha_{i_2} = \dots = \alpha_{i_m} = 0$. This implies that \bar{C} coincides with C , which is a contradiction. QED

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